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Distribution of eigenvalues near the boundary of essential spectrum

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We study in this note the asymptotic distribution of discrete eigenvalues (bound states) near the boundary of essential spectrum for Schrodiger operators and Dirac operators.

1. Schrodinger operators

Let us consider the following eigenvalue problem

$$(1.1) \qquad -\Delta u - p(x)u = \lambda u \qquad \text{in } L^2(R^n)$$

If $p(x)$ does not decay at infinity too rapidly, then the operator $-\Delta - p(x)$ has an infinite sequence of negative eigenvalues approaching to zero. We denote by $n(r)$ $(r > 0)$ the number of eigenvalues less than $-r$ of problem (1.1). We are concerned with the asymptotic behavior of $n(r)$ as r tends to zero.

This problem was studied in Brownell and Clark (2), and McLeod (3) under the condition that a potential $p(x)$ is sufficiently close to a spherically symmetric potential and non-negative. The purpose of this note is to study the distribution of eigenvalues by the different method without assuming the above condition.
We impose the following assumption on potentials $p(x)$;

(1.2)
(1) $p(x)$ is decomposed as $p(x) = p_1(x) + p_2(x)$;

(2) $p_1(x)$ is a smooth function satisfying

$$\lim_{|x|\to\infty} |x|^m p_1(|x|\omega) = a(\omega) \qquad x = |x|\,\omega \qquad \omega \in S^{n-1}$$

where $a(\omega)$ is a function not necessarily positive defined on S^{n-1} (the n-1 dimensional unit sphere);

(3) $p_2(x)$ is a non-negative and integrable function with compact support, if $n \leq 2$;

(4) $p_2(x)$ belongs to $L^{\frac{n}{2}}$, if $n > 2$.

Theorem 1. Let $p(x)$ be a potential satisfying the above assumption (1.2) and suppose that $m < 2$. Then, we have

$$n(r) = C\, r^{\frac{n}{2}-\frac{n}{m}} + o(r^{\frac{n}{2}-\frac{n}{m}}) \tag{1.3}$$

where $C = (2\pi)^{-n} \frac{\sigma_{n-1}}{2n} \frac{\Gamma[\frac{n}{2}]\Gamma[\frac{n}{m}-\frac{n}{2}]}{\Gamma[\frac{n}{m}]} \int_{S^{n-1}} a_+(\omega)^{\frac{n}{m}} d\omega$, σ_{n-1} is the surface measure of S^{n-1}, and $a_+(\omega) = \max(a(\omega), 0)$.

2. Dirac operators.

We consider the following eigenvalue problem

$$S\varphi = \left(\sum_{k=1}^{3} \alpha_k \xi_k + \alpha_4\right)\varphi - p(x)\varphi = \lambda\varphi \qquad \text{in } L^2(R^3)^4 \tag{2.1}$$

Here $\xi_k = -i\left(\frac{\partial}{\partial x_k}\right)$ ($k = 1,2,3$); $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_4)$ is a four-component function; α_k $(k = 1,2,3,4)$ are the Dirac numerical matrices satisfying the relationship $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}$; $p(x)$ is a scalar potential. Throughout this section, for the sake of simlicity we assume that $p(x)$ is a smooth positive function satisfying

$$\lim_{|x|\to\infty} |x|^m p(|x|\,\omega) = a(\omega).$$

We denote by $n(r)$ the number of eigenvalues lying in $(-1, 1-r)$ of problem (2.1). Since we assume that $p(x)$ discrete eigenvalues cannot admit the $\lambda = -1$ as a cluster point.

Theorem 2. Let p(x) be a scalar potential with the above condition. Then we have

$$(2.2) \qquad n(r) = C\, r^{\frac{3}{2}-\frac{n}{m}} + o\,(r^{\frac{3}{2}-\frac{n}{m}}),$$

where we can give the explicit expression for the constant C but we do not refer to it.

3. Sketch of the proof.

Sketch of the proof of Theorem 1:

Our method is based on the fact that n(r) coincides with the maxmal dimension of subspaces lying in $C_0^\infty(R^n)$, the set of smooth functions with compact support, such that

$$(-\Delta u,\ u) - (p(x)u,\ u) < -r\ (u,\ u),$$

where (,) means the usual inner product in $L^2(R^n)$.
From the above fact, we see that n(r) is equal to the number of positive eigenvalues less than one of the problem

$$(3.1) \qquad -\Delta u + r\, u = \lambda p(x)\, u.$$

Putting $r = \frac{1}{h}$ ($h \to \infty$) , we have

$$(3.2) \qquad -h\Delta u + u = \lambda p(x)\, u \quad (\ 0 < \lambda < h)\ .$$

We denote by $N_h(\lambda)$ the number of positive eigenvalues less than λ. Then, we have

Theorem 3. Let $p_1(x)$ be a smooth positive potential defined by (1.2) ($a(\omega) > 0$). Then there exist $C_1(\delta)$ and $C_2(\delta)$ for each fixed $\delta > 0$ (small enough) such that

$$\left| N_h(\lambda) - C\, h^{-\frac{n}{2}}\lambda^{\frac{n}{m}} \right| \leq \delta\, h^{-\frac{n}{2}}\lambda^{\frac{n}{m}}$$

$$\text{for } \lambda \geq \max(\ C_1(\delta),\ C_2(\delta)\ h^{\alpha}\ ) \quad (0<\alpha<1)$$

where C is the constant defined by Theorem 1.
The involved part of the proof of Theorem 3 is to show that $0 < \alpha < 1$. Thus we can get the desired result for n(r) when a potential p(x) is smooth and positive.

We need some lemmas in order to extend the result obtained above to the case of potentials with singularities.

Let us define some operators as follows:

$$(3.3)\qquad T(r) = (-\Delta + r)^{-1}(p_1 + p_2)\ ,$$

$$T_1(r) = (-\Delta + r)^{-1}\ p_1 \quad \text{and} \quad T_2(r)\quad (-\Delta + r)^{-2}\ p_2\ .$$

The operators $T(r)$, $T_1(r)$ and $T_2(r)$ defined above are compact self-adjoint operators acting in $H^1(R^n)$, the usual Sobolev space of order one, with the scalar product

$$(u,\ v)_r = ((-\Delta + r)^{1/2}u,\ (-\Delta + r)^{1/2}v) \quad \text{for} \quad u, v \in H^1(R^n)\ .$$

Lemma 3.2 Let $p_2(x)$ be a potential satisfying (3) or (4) of the assumption (1.2) . $M(\lambda)$ denotes the number of eigenvalues less than λ of the following problem:

$$(-\Delta + r)u = \lambda\, p_2(x)\ u\ , \qquad u \in H^1(R^3)\ . \quad \text{Then}$$

$$(3.4)\qquad M(\lambda) = C\int_{R^n} p_2(x)^{n/2}dx\ \lambda^{n/2} + o(\lambda^{n/2})\ ,$$

where C is the absolute constant and the remainder estimate holds uniformly with respect to r ($1 \geq r \geq 0$).

The above lemma was proved in [1] and [4] .

Lemma 3.3 There is a constant $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, $T_2(r)$ has at least one eigenvalue in $(\varepsilon/3,\ \varepsilon/2)$.

Lemma 3.4 Let $m(r, \varepsilon)$ be the number of eigenvalues greater than ε of $T_2(r)$. Then there exists a constant $C(\varepsilon)$ independent of r such that

$$(3.5) \qquad m(r,\varepsilon) \leq C(\varepsilon)$$

Lemma 3.5 For any ε 0 (small enough), there is a constant $r(\varepsilon)$ such that for any $r < r(\varepsilon)$, $T_1(r)$ has at least one eigenvalue in $(1 - 2\varepsilon, 1 - \varepsilon)$.

Lemmas 3.3, 3.4 and 3.5 are verified with the aid of Lemma 3.2 .

By means of Lemmas 3.3, 3.4 and 3.5, we see that

$$(3.6) \quad \limsup_{r\to 0} r^{n/m-n/2}\, n(r) \leq \limsup_{r\to 0} r^{n/m-n/2}\, n(r,(1-2\varepsilon)^{-1}p_1)$$

where $n(r,p_1)$ is the number of eigenvalues less than $-r$ of the problem, $-\Delta u - p_1(x)\, u = \lambda u$.

On the other hand , it is not difficult to show that

$$(3.7) \quad \liminf_{r\to 0} r^{n/m-n/2}\, n(r) \geq \liminf r^{n/m-n/2}\, n(r,p_1).$$

Since ε is arbitrary , by combining (3.7) with (3.6) we obtain the proof of Theorem 1 .

Sketch of the proof of Theorem 2.

Lemma 3.6 For each fixed δ 0, there exist operators $A(\delta)$ and $B(\delta)$ such that for $\varphi \in [C_0^\infty(R^3)]^4$,

$$(3.8)\quad [B(\delta)\varphi, \varphi] \leq [(S^2-I)\varphi, \varphi] \leq [A(\delta)\varphi, \varphi],$$

where

$$A(\delta) = \begin{pmatrix} E(\delta) & & & 0 \\ & E(\delta) & & \\ & & F(\delta) & \\ 0 & & & F(\delta) \end{pmatrix} \quad \text{and} \quad B(\delta) = \begin{pmatrix} G(\delta) & & & 0 \\ & G(\delta) & & \\ & & H(\delta) & \\ 0 & & & H(\delta) \end{pmatrix},$$

while

$E(\delta) = (1 + \delta)(-\Delta) - 2p(x) + C(\delta)p(x)^2$,

$F(\delta) = (1 + \delta)(-\Delta) + 2p(x) + C(\delta)p(x)^2$,

$G(\delta) = (1 - \delta)(-\Delta) - 2p(x) - C(\delta)p(x)^2$,

$H(\delta) = (1 - \delta)(-\Delta) + 2p(x) - C(\delta)p(x)^2$, and $[\varphi, \psi]$ means the scalar product in $[L^2(R^3)]^4$.

From Lemma 3.6, by applying the same argument as in the proof of Theorem 1 to the operators $E(\delta)$ and $G(\delta)$, Theorem 2 follows.

The detailed proofs of Theorem 1 and Theorem 2 will be announced in [5], [6].

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